

Dressing Technique for Intermediate Hierarchies

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50009 Zaragoza, Spain***Abstract**

A generalized AKNS systems introduced and discussed recently in [1] are considered. It was shown that the dressing technique both in matrix pseudo-differential operators and formal series with respect to the spectral parameter can be developed for these hierarchies.

1 Introduction

It was shown [1] that besides classical examples of the hierarchies of the integrable equations corresponding to the principal and homogeneous pictures of the basic representation of the affine Lie algebras it is possible to consider intermediate cases. These cases are associated with the different choices of a Heisenberg subalgebra in a loop algebra \tilde{g} which in turn correspond to the different choices of conjugacy class in the Weyl group [2]. The principal and homogeneous pictures correspond to the classes defined by Coxeter and identity elements respectively.

The aim of this paper is to show that for these intermediate hierarchies the dressing technique in terms of matrix pseudo-differential operators and formal series with respect to spectral parameter can be developed as it was done for usual generalized KdV_n hierarchies (see [3] and reference therein). The starting point of our consideration will be a Hamiltonian symmetry reduction and a construction of the vector field tangent to the reduced manifold and related to the integrable flows of the corresponding hierarchy. Solving the tangency

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constraints we will find an explicit expression for the gradient of the gauge-invariant functional (20) which ensures the tangency condition. We will also prove that dressing leads to integrable hierarchies defined by this gradient. These results were utilized previously in the series of papers [4], where the nonlocal partners to the integrable hierarchies associated with the principal realization of \tilde{sl}_2 were considered.

All results in the paper is formulated for the nontwisted loop algebra \tilde{sl}_n . The generalization of this approach to the hierarchies corresponding to BCD series requires the solution to the problem of classification the regular elements in the inequivalent Heisenberg subalgebras of the corresponding loop algebras. This problem was considered in [5].

The paper is organized as follows. In the subsection 1.1 we list the regular elements of the Heisenberg subalgebras of \tilde{sl}_n which we will use in our construction. Section 2 is devoted to the Hamiltonian reduction and solution the tangency constraints. In the section 3 dressing by matrix pseudo-differential operators and formal series with respect to the spectral parameter is discussed and connection to the integrable hierarchies obtained by the Hamiltonian reduction is established.

1.1 Regular elements of the Heisenberg subalgebras of $\widehat{\tilde{sl}_n}$

It was shown in [2] that the basic representations of the affine Lie algebras of the ADE type can be parametrized by a set of the representatives of the conjugacy classes of the corresponding Weyl group W . One can associate with each representative $w \in W$ the Heisenberg subalgebra s_w of the untwisted loop algebra \tilde{g} . The choice $w = 1$ or $w = \text{Coxeter element}$ corresponds to the homogeneous or principal construction of the basic representation. In former case the homogeneous Heisenberg subalgebra is $\tilde{h} = \mathbb{C}[\lambda, \lambda^{-1}] \otimes h$, where h is Cartan subalgebra of the underlying simple Lie algebra g . In later case (for $\tilde{g} = \tilde{sl}_n$ Kac-Moody algebra) the principal Heisenberg subalgebra is the linear span of the elements

$$a_{k,m} = \lambda^k \begin{pmatrix} & & \lambda \\ & 1 & \\ & \ddots & \\ & & 1 \end{pmatrix}^m, \quad m, k \in \mathbb{Z}, \quad 1 \leq m \leq n-1. \quad (1)$$

The explicit construction of the all inequivalent graded Heisenberg subalgebras of the algebra \tilde{sl}_n was proposed in [6]. For this algebra the Weyl group is isomorphic to the symmetric group S_n . The conjugacy classes in S_n and hence the inequivalent Heisenberg subalgebras of \tilde{sl}_n are parametrized by partition of n ; $\underline{n} = n_1 + n_2 + \dots + n_p$, $n_1 \geq n_2 \geq \dots \geq n_p \geq 1$. The Heisenberg subalgebra $s_{\underline{n}}$ is spanned by the elements

$$\Lambda = \begin{pmatrix} d_1 \Lambda_{n_1}^{\ell_1} & & & \\ & d_2 \Lambda_{n_2}^{\ell_2} & & \\ & & \ddots & \\ & & & d_p \Lambda_{n_p}^{\ell_p} \end{pmatrix}, \quad \Lambda_{n_i} = \begin{pmatrix} & & \lambda \\ & 1 & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad (2)$$

where Λ_{n_i} are $n_i \times n_i$ matrices, ℓ_i are arbitrary integers and d_i are arbitrary *distinct* complex numbers, $i = 1, \dots, p$.

Examining the eigenvalues of the matrix Λ one can prove [7] that the regular graded elements exist only in those Heisenberg subalgebras of \tilde{sl}_n which belong to the special partitions. Namely, the “equal block partition” $n = pr$ and “equal block plus singlet partition” $n = pr + 1$ ($n_i = r > 1$, $\ell_i = \ell$ is relative prime to r , $i = 1, \dots, p$). In the case $n = pr$ after reordering of the basis the element Λ can be written in the form

$$\Lambda = \lambda^m \Lambda_r^\ell \otimes D, \quad D = \text{diag}(d_1, d_2, \dots, d_p), \quad \sum_{i=1}^p d_i = 0, \\ 1 \leq \ell \leq r-1, \quad y_i \neq 0, \quad y_i^r \neq y_j^r, \quad i \neq j, \quad m \in \mathbb{Z}. \quad (3)$$

The element Λ is of grade $(\ell + mr)$ with respect to the grading operator $r\lambda \frac{d}{d\lambda} + \text{ad}H$, where the diagonal matrix H is given by

$$H = \text{diag}[\underbrace{H_r, H_r, \dots, H_r}_{p \text{ times}}], \quad \text{where} \quad H_r = \text{diag}[j, j-1, \dots, -j], \quad j = \frac{r-1}{2}. \quad (4)$$

As we mentioned it was suggested in [1] to use the graded regular elements of the inequivalent Heisenberg subalgebras of the loop algebra \tilde{sl}_n to obtain the generalizations of the Drinfeld-Sokolov hierarchies [3]. In the following sections we will consider Hamiltonian reduction and dressing technique based on the grade 1 regular element

$$\Lambda = \Lambda_{r,p} = \Delta_r \otimes D = I + \lambda e$$

of the equal block partition. It corresponds to the type-I hierarchies in terms of the papers [1]. For the integrable hierarchies associated with the partition $n = pr + 1$ see [5].

2 Hamiltonian Reduction

In the first paragraph of this section we will sketch the main steps of the Hamiltonian reduction for the system associated with the loop algebra \tilde{sl}_n . The details can be found elsewhere (see for example [3]).

Let M be the space of differential operators $\mathcal{L} = \partial + J$, where $J = q(x) - \lambda e$ is the special element of the loop algebra \tilde{sl}_n , ∂ is derivative with respect to the parameter $x \in S^1$ and the function $q(x)$ defines a map $q(x): S^1 \rightarrow sl_n$. Following the arguments of the papers [3, 8] one can show that the space M is Poisson space with two natural compatible Poisson brackets. The action of $\text{ad} H$ induces the decomposition of $\tilde{sl}_n = \tilde{sl}_n^+ \oplus \tilde{sl}_n^0 \oplus \tilde{sl}_n^-$, where \tilde{sl}_n^\pm and \tilde{sl}_n^0 are block upper, block lower and block diagonal matrices. Transformation $\mathcal{L} \rightarrow e^s \mathcal{L} e^{-s}$, $s \in \tilde{sl}_n^+$ preserve both Poisson brackets as well as monodromy invariants of the operator \mathcal{L} . First step of the reduction is imposing of the constraint $q^- = -I$. This special choice of the matrix q corresponds to the fixing of the image of the momentum map induced by the Hamiltonian group action $e^s \mathcal{L} e^{-s}$. Second step is a factorization of the constrained space $M^{\text{con}} = M|_{q^- = -I}$ by the adjoint action of the group $S = e^s$

$$\mathcal{L}^{\text{con}} \rightarrow S \mathcal{L}^{\text{con}} S^{-1}, \quad q - \Lambda \rightarrow S(q - \Lambda) S^{-1} - \partial S \cdot S^{-1}, \quad (5)$$

where now $\mathcal{L}^{\text{con}} = \partial + q - \Lambda_{r,p}$ and $q \in \tilde{sl}_n^+ \oplus \tilde{sl}_n^0$. The action (5) means that the space M^{con} is an orbit of the adjoint action of the group S . It is well known [3] that it is possible to separate different orbits by introducing the r -order (matrix in our case) differential operator L

$$L = D^{-r} \partial^r + \sum_{i=0}^{r-1} u_i \partial^i, \quad (6)$$

where u_i are $p \times p$ matrices with entries which are differential polynomials of the elements of the matrix q . To obtain exact expression for the operator L we have to consider linear problem $\mathcal{L}\Psi = 0$ for the column Ψ and exclude first $n - p$ components of Ψ . The remaining p components obey an equation $\sum_{j=n-p+1}^n L_{ij} \Psi_j = \lambda \Psi_i$, $i = n - p + 1, \dots, n$. The group element S acts on the column Ψ by left multiplication $\Psi \rightarrow S\Psi$, so the last p components of the column Ψ do not change under this transformation. It means that the operator L is indeed invariant with respect to transformation (5) and hence can be chosen to parametrize the reduced factor space $\overline{M} = M^{\text{con}}/S$.

There exists more formal way to obtain the operator L . Following [3] let us write operator \mathcal{L} in the form

$$\mathcal{L} = \begin{pmatrix} \alpha & \beta \\ A & \gamma \end{pmatrix}, \quad (7)$$

where α , β , γ and A are the $p \times (n - p)$, $p \times p$, $(n - p) \times p$ and $(n - p) \times (n - p)$ matrices respectively. Then $p \times p$ matrix differential operator L can be obtained by the formula

$$L = D^{-1}(\beta - \alpha A^{-1} \gamma). \quad (8)$$

Let us fix the gauge for the operator \mathcal{L} . We choose the following canonical gauge

$$\mathcal{L}^{\text{can}} = \partial + q^{\text{can}} - \Lambda_{r,p}, \quad \text{where} \quad q^{\text{can}} = \begin{pmatrix} v_{r-1} & v_{r-2} & \cdots & v_1 & v_0 \\ & & & 0 & \end{pmatrix}, \quad (9)$$

and v_i are $p \times p$ matrices. By this choice the relations between matrices u_i and v_i , $i = 0, \dots, r - 1$ have the simplest form

$$u_i = D^{-1} v_i D^{-i}. \quad (10)$$

In the sequel we will need only a second Poisson bracket which in the space M^{con} can be defined as follows. For two functionals $f(q)$ and $g(q)$ associated with the operator \mathcal{L} define the Poisson bracket

$$\{f, g\}(q) = \int_{S^1} \text{tr} (\partial + q - I) [\text{grad}_q f, \text{grad}_q g] dx, \quad (11)$$

where $\text{grad}_q g$ is given by

$$\left. \frac{dg(q + \varepsilon h)}{d\varepsilon} \right|_{\varepsilon=0} = \text{tr} (\text{grad}_q g \cdot h). \quad (12)$$

Define the gauge invariant functional on the space M^{con} by the formula

$$\ell_X = \int_{S^1} \text{tr} \text{res} (LX) dx, \quad (13)$$

where X is $p \times p$ matrix pseudo-differential operator of the form

$$X = \sum_{j=1}^{\infty} \partial^{-j} \circ X_j \quad (14)$$

and res means the coefficient at ∂^{-1} . Using the formulas (6),(8),(12) and arguments of the paper [3] the matrix elements of the gradient of the functional ℓ_X read as

$$(\text{grad}_q \ell_X)_{ij} = \text{res} (D^{i-r} \partial^{r-i} X (L \partial^{j-r-1})_+ D^{r-j}), \quad i, j = 1, \dots, r, \quad (15)$$

where indices i, j numerate the $p \times p$ blocks of the gradient. Upper triangular blocks of the $\text{grad}_q \ell_X$ are not determined by (12) and Poisson bracket (11) on the gauge invariant functionals does not depend on the choice of the upper triangular part of the $\text{grad}_q \ell_X$ and are correctly defined.

Using (15) one can easily calculate the vector field $[\partial + q^{\text{can}} - I, \text{grad}_q \ell_X]$. The nonzero entries of this field are

$$\begin{aligned} ([\partial + q^{\text{can}} - I, \text{grad}_q \ell_X])_{1j} &= D(LX)_- L \partial^{j-r-1} D^{j-r}, \\ ([\partial + q^{\text{can}} - I, \text{grad}_q \ell_X])_{1r} &= D[(LX)_- L \partial^{-1} - (\partial D^{-1})^{r-1} (XL)], \\ ([\partial + q^{\text{can}} - I, \text{grad}_q \ell_X])_{ir} &= -(\partial D^{-1})^{r-i} XL, \end{aligned} \quad (16)$$

where $j = 1, \dots, r-1$ and $i = 2, \dots, r$. It is seen from (16) that this vector field is tangent to M^{con} but is not tangent to reduced manifold \overline{M} . The condition of tangency yields the $r-1$ constraints for the operator X . Indeed, the conditions $\text{res} XL = \text{res} \partial D^{-1} XL = \text{res} (\partial D^{-1})^{r-2} XL = 0$, which are equivalent to

$$\text{res} XL = \text{res} \partial XL = \text{res} \partial^{r-2} XL = 0, \quad (17)$$

allow one to express matrix coefficients X_{r+1}, \dots, X_{2r-1} of the operator X via the coefficients X_1, \dots, X_r and matrix coefficients of the operator L .

Using the identity [3, 9]

$$[L(XL)_-]_+ = \sum_{a=1}^r (L \partial^{-a})_+ \circ \text{res} (\partial^{a-1} XL) \quad (18)$$

which is valid for any differential and pseudo-differential operators L and X one can show that if operator X satisfy (17), than only zero coefficient of the differential operator $[L(XL)_-]_+$ is nonzero

$$[L(XL)_-]_0 = \text{res} (D^{-r} \partial^{r-1} XL).$$

$[B]_i$ means the coefficient at ∂^i of the operator B . Thus the nonzero entries of the vector field (16) after solution of the constraints (17) are

$$([\partial + q^{\text{can}} - I, \text{grad}_q \ell_X] \Big|_{\text{const}=0})_{1j} = D[(LX)_- L - L(XL)_-] \partial^{j-r-1} D^{j-r}, \quad (19)$$

where $j = 1, \dots, r$. It is possible to solve the constraints (17) in the expression for the gradient (15). Indeed, for $j > i$ we have

$$\begin{aligned} \text{res}(D^{i-r}\partial^{r-i}X(L\partial^{j-r-1})_+D^{r-j}) &= \text{res}(D^{i-r}\partial^{r-i}XL\partial^{j-r-1}D^{r-j}) - \\ &- \text{res}(D^{i-r}\partial^{r-i}X(L\partial^{j-r-1})_-D^{r-j}), \end{aligned}$$

and second summand in this formula does not depend on the matrices X_{r+1}, \dots, X_{2r-1} . Using the obvious identity

$$\begin{aligned} \text{res}\partial^{r-i}XL\partial^{j-r-1} &= \sum_{a=0}^{r-1} C_{a+r-i}^{a+j-i} \partial^{a+j-i} \text{res}(\partial^{-a-1}XL) + \\ &+ \sum_{b=1}^{j-i} C_{r-i-b}^{j-i-b} \partial^{j-i-b} \text{res}(\partial^{b-1}XL) \end{aligned}$$

the expression for the $(\text{grad}_q \ell_X|_{\text{constr}=0})_{ij} = (\text{Grad}_q \ell_X)_{ij}$ is

$$(\text{Grad}_q \ell_X)_{ij} = \begin{cases} \text{res}(D^{i-r}\partial^{r-i}X(L\partial^{j-r-1})_+D^{r-j}), & j \leq i \\ -\text{res}(D^{i-r}\partial^{r-i}X(L\partial^{j-r-1})_-D^{r-j}) + \\ + D^{i-r} \sum_{a=0}^{r-1} C_{a+r-i}^{a+j-i} \partial^{a+j-i} \text{res}(\partial^{-a-1}XL) D^{r-j}, & j > i, \end{cases} \quad (20)$$

where C_a^b are binomial coefficients and $1 \leq i, j \leq p$ numerate the blocks of $rp \times rp$ matrix $\text{Grad}_q \ell_X$.

From (19) and (15) follow that Poisson bracket (11) for the functionals ℓ_X and ℓ_Y coincides with the second Gelfand-Dickey bracket [10]

$$\{\ell_X, \ell_Y\} = \int_{S^1} \text{tr} \text{res} [(LX)_-LY - YL(XL)_-]. \quad (21)$$

The auxiliary linear problem for the operator L is the linear problem for the p -component KdV $_r$ hierarchy studied previously in [11, 12, 13].¹ In [7] the monodromy invariants for the operator L were calculated. Let us briefly formulate this result.

Since all entries of the diagonal matrix D^{-r} are different, it is possible to find the operator of the form $g = \mathbf{1}_p + \sum_{i=1}^{\infty} g_i \partial^{-i}$ using the recursion procedure such that

$$\hat{L} = g^{-1}Lg = D^{-r}\partial^r + \sum_{i=1}^{\infty} a_i \partial^{r-i}, \quad (22)$$

where a_i are the diagonal matrices. The matrices g_i can be uniquely defined from (22) by requirement that all $\text{diag } g_i = 0$ [3, 7]. The involutive set of the monodromy invariants for the operator L are

$$\mathcal{H}_{0,i} = \int_{S^1} (D^r u_{r-1})_{ii}, \quad \mathcal{H}_{k,i} = \int_{S^1} \text{res} \left(\hat{L}^{k/r} \right)_{ii}, \quad i = 1, \dots, p; \quad k = 1, 2, \dots, \quad (23)$$

and fractional power of the operator \hat{L} is pseudo-differential operator with the property $(\hat{L}^{1/r})^r = \hat{L}$. In case of p -component KdV $_r$ the Hamiltonians $\mathcal{H}_{0,i}$ are absent.

¹The requirement $\text{diag } u_{r-1} = 0$ has to be imposed in this case on the operator L .

3 Dressing Technique for the p -Component KdV $_r$ Hierarchies

3.1 Dressing by matrix pseudo-differential operators

The aim of this subsection is the formulation of the Zakharov-Shabat dressing technique applied to the $p \times p$ matrix differential operator L of the r th order. We will proceed in a way similar to the standard scalar dressing procedure for the usual KdV $_n$ hierarchies.

Let us consider the operators $L = (\Delta\partial)^r + \sum_{i=0}^{r-1} u_i \partial^i$, $L_0 = (\Delta\partial)^r$ and columns ψ , $\overset{\circ}{\psi}$ which are the solutions to the corresponding linear problems

$$L\psi = \lambda\psi, \quad L_0\overset{\circ}{\psi} = \lambda\overset{\circ}{\psi}, \quad \Delta = D^{-1} = \text{diag}(\Delta_1, \dots, \Delta_p). \quad (24)$$

Suppose that solutions ψ and $\overset{\circ}{\psi}$ are connected by invertible matrix valued operator Volterra $K(x, \partial)$

$$\psi = K(x, \partial)\overset{\circ}{\psi} = \left(\mathbf{1}_p + \sum_{j=1}^{\infty} K_j(x) \partial^{-j} \right) \overset{\circ}{\psi}. \quad (25)$$

Equation (24) yields the recurrence relation for the matrix coefficients of the operator K

$$\begin{aligned} [K_{i+r}, \Delta^r] &= \sum_{k=0}^{r-1} \frac{1}{k!} \left(\frac{\partial^k L}{\partial \partial^k} \right) \cdot K_{i+k}, \\ i &= 1-r, 2-r, \dots, \quad K_0 = 1, \quad K_j = 0, \quad j < 0, \end{aligned} \quad (26)$$

where $(\partial^k L / \partial \partial^k)$ means the k -fold formal derivative of the differential operator L with respect to the symbol ∂ and $(B) \cdot K_i(x)$ means the action of the differential operator B applied to the function $K_i(x)$. For example, first two equations from the set (26) are

$$[K_1, \Delta^r] = u_{r-1}, \quad (27)$$

$$[K_2, \Delta^r] = r\Delta^r K_1' + u_{r-1}K_1 + u_{r-2}, \quad (28)$$

K_i' means the derivative with respect to x .

Lemma 1. *If $\text{diag } u_{r-1} = 0$ the recurrence relations (26) for the matrix functions K_i can be solved.*

Proof. Since the entries Δ_i^r of the diagonal matrix Δ^r are all different it is easy to see that diagonal part of the equation (26) determines the diagonal elements of the matrix $(K_{i+r-1})_{jj}$, while the off-diagonal part of the same equation yields the off-diagonal elements of the matrix $(K_{i+r})_{jk}$, $j \neq k$. For example, the off-diagonal elements of the matrix K_1 defined by (27) and the diagonal elements of the same matrix can be obtained from (28)

$$\begin{aligned} (K_1)_{kj} &= \frac{(u_{r-1})_{kj}}{\Delta_j^r - \Delta_k^r}, \quad k \neq j, \\ r\Delta_i^r (K_1)'_{jj} &= \sum_{k=1}^p \frac{(u_{r-1})_{jk}(u_{r-1})_{kj}}{\Delta_k^r - \Delta_j^r} - (u_{r-2})_{jj}, \quad j = 1, \dots, p. \end{aligned} \quad (29)$$

To define the time evolution of the operator L we introduce $p \times p$ matrices $E_a = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the a th place and infinite set of the time variables $t_s^{(a)}$, $a = 1, \dots, p$, $s = 1, 2, \dots$. Let us define the time evolution of the operator K and the solution ψ of the initial linear problem by

$$\frac{\partial K}{\partial t_s^{(a)}} K^{-1} + (K(\Delta\partial)^s E_a K^{-1})_- = 0, \quad \frac{\partial \psi}{\partial t_s^{(a)}} = (\Delta\partial)^s E_a \psi. \quad (30)$$

The following theorem is valid.

Theorem 2. *The compatibility condition for the linear problems $L\psi = \lambda\psi$ and $\partial_{t_s^{(a)}}\psi = (K(\Delta\partial)^s E_a K^{-1})_+\psi$*

$$\frac{\partial L}{\partial t_s^{(a)}} = (LX)_+ L - L(XL)_+, \quad \text{where } X = (K(\Delta\partial)^{s-r} E_a K^{-1})_- \quad (31)$$

is equivalent to the evolution equation for the matrix q^{can} defined by the bracket (11) with Hamiltonian $\mathcal{H}_{s,a}$ from the set (23).

The proving of this theorem is based on the following Lemma

Lemma 3. *The operator $A = g^{-1}K$, where operators K and g defined by (26) and (22) respectively, is diagonal pseudo differential operator.*

Proof. Suppose that operator A has nonzero off-diagonal part and define the column vector $\hat{\psi} = g^{-1}\psi = A\psi$. It satisfies the linear problem $\hat{L}\hat{\psi} = \lambda\hat{\psi}$, where \hat{L} is diagonal operator defined by (22). Now the relations $\hat{L}A = AL_0$ and $\Delta_i^r \neq \Delta_j^r$, $i \neq j$ yield that all off-diagonal entries of the operator A are identically zero.

The statement of the theorem 2 follows now from the equation (19) and the relation

$$\begin{aligned} \ell_X &= \int_{S^1} \text{tr res}(LX) dx \\ &= \int_{S^1} \text{tr res}(K(\Delta\partial)^r K^{-1})(K(\Delta\partial)^{s-r} E_a K^{-1})_- dx \\ &= \int_{S^1} \text{tr res} K(\Delta\partial)^s K K^{-1} E_a K^{-1} dx \\ &= \int_{S^1} \text{tr res}(g^{-1}K(\Delta\partial)^s K^{-1}g)(g^{-1}K E_a (g^{-1}K)^{-1}) dx \\ &= \int_{S^1} \text{tr res}(\hat{L}^{s/r} E_a) dx = \int_{S^1} \text{res}(\hat{L}^{s/r})_{aa} dx = \mathcal{H}_{s,a}. \end{aligned} \quad (32)$$

In the second line of (32) we used the fact that L is differential operator and the operator $\hat{L}^{1/r} = g^{-1}K(\Delta\partial)(g^{-1}K)^{-1}$ obviously satisfies the relation $(\hat{L}^{1/r})^r = \hat{L}$.

It follows from the definition of the time flows that $\partial/\partial x = \sum_{a=1}^p d_a \partial/\partial t_1^{(a)}$, $d_a = \Delta_a^{-1}$. Indeed,

$$\sum_{a=1}^p y_a \frac{\partial \psi}{\partial t_1^{(a)}} = (K \partial K^{-1})_+ \psi = \frac{\partial \psi}{\partial x}. \quad (33)$$

Moreover it can be shown that operator L and hence \mathcal{L}^{can} does not depend on the certain combination of the times $t_s^{(a)}$. From (23) and (24) follows that

$$\sum_{a=1}^p \mathcal{H}_{mr,a} = 0, \quad m = 1, 2, \dots \quad (34)$$

and hence

$$\sum_{a=1}^p \frac{\partial L}{\partial t_{mr}^{(a)}} = 0. \quad (35)$$

Except continuum times $t_s^{(a)}$, $s \geq 1$ we can introduce p discrete times $t_0^{(a)}$, $a = 1, \dots, p$. The evolutions of column $\overset{\circ}{\psi}$ and operator K with respect to these times are given by²

$$\begin{aligned} \overset{\circ}{\psi}(t_0^{(a)} + \delta_{ab}) &= (\Delta\partial)E_b\overset{\circ}{\psi}, \\ [K(t_0^{(a)} + \delta_{ab})(\Delta\partial)E_bK^{-1}]_- &= 0, \quad a, b = 1, \dots, p. \end{aligned} \quad (36)$$

The compatibility conditions of the linear problems

$$\begin{aligned} \psi(t_0^{(a)} + 1) &= [K(t_0^{(a)} + 1)(\Delta\partial)E_aK^{-1}]_+ \psi = [(\Delta\partial)E_a + m_a] \psi, \\ \frac{\partial \psi}{\partial t_0^{(b)}} &= [K(\Delta\partial)^s E_b K^{-1}]_+ \psi, \end{aligned} \quad (37)$$

yield the p -component analog of the modified KdV _{r} hierarchy. Indeed, in the simplest case $p = 1$, $r = 2$ and $s = 3$ this compatibility condition reads

$$\begin{aligned} \frac{\partial m}{\partial t_3} &= [K(t_0 + 1)\partial^3 K(t_0 + 1)^{-1}]_1 m' - [K(t_0)\partial^3 K(t_0)^{-1}]'_0 \\ &\quad - m [K(t_0 + 1)\partial^3 K(t_0 + 1)^{-1} - K(t_0)\partial^3 K(t_0)^{-1}]_0 \\ &= \frac{1}{4}m''' - \frac{3}{2}m^2 m', \end{aligned}$$

where $K(t_0 + 1)$, $K(t_0)$ are defined by

$$\begin{aligned} K(t_0 + 1)\partial^2 K(t_0 + 1)^{-1} &= \partial^2 + u(t_0 + 1) = (\partial + m)(\partial - m) \\ K(t_0)\partial^2 K(t_0)^{-1} &= \partial^2 + u(t_0) = (\partial - m)(\partial + m). \end{aligned}$$

From the discrete evolutions (36) follows that the operator $\mathcal{K} = \sum_{a=1}^p K(t_0^{(a)} + r)E_a$ will satisfies the relation

$$(\mathcal{K}(\Delta\partial)^r K^{-1})_- = 0. \quad (38)$$

This equation can be solved if we impose the restriction

$$\mathcal{K} = \sum_{a=1}^p K(t_0^{(a)} + r)E_a = K \quad (39)$$

²In what follows we will write the dependence on the discrete times explicitly only if they are changed and expression $K(t_0^{(a)} + 1)$ will mean the shift of discrete time $t_0^{(a)}$ in the coefficients of the operator K by 1 for fixed a .

which is nothing but reduction from p -component KP hierarchy to the p -component KdV $_r$ hierarchy expressed in terms of the dressing operators and this reduction corresponds to the equal block gradation of the loop algebra \widetilde{sl}_n described above. In the case of one-component theory the relation (39) shows that corresponding τ -function is periodic function of discrete time with period equal to the order of the differential operator L . The relation (39) yields the generalized Miura transformation

$$\begin{aligned} \sum_{a=1}^p \left((\Delta\partial)E_a + m_a(t_0^{(a)} + r - 1) \right) \dots \left((\Delta\partial)E_a + m_a(t_0^{(a)} + 1) \right) \left((\Delta\partial)E_a + m_a(t_0^{(a)}) \right) \\ = (\Delta\partial)^r + \sum_{i=0}^{r-1} u_i \partial^i, \end{aligned} \quad (40)$$

where

$$m_a = K_1(t_0^{(a)} + 1)\Delta E_a - E_a\Delta K_1(t_0^{(a)}) \quad (41)$$

and K_1 is the coefficient at the ∂^{-1} in the operator K . Generalized Miura transformation (40) respect the condition $\text{diag}(u_{r-1}) = 0$. Indeed,

$$\begin{aligned} \text{diag}(u_{r-1}) &= \text{diag} \sum_{a=1}^p \left[K_1(t_0^{(a)} + r)\Delta^r E_a - E_a\Delta^r K_1(t_0^{(a)}) \right. \\ &\quad \left. + \dots + K_1(t_0^{(a)} + 1)\Delta^r E_a - E_a\Delta^r K_1(t_0^{(a)} + 1) \right] = 0 \end{aligned}$$

due to (39) and (27). The relation (39) means that shifts on r with respect to each discrete times are not independent and this relation is the discrete counterpart of the relation (35).

We will show in the next subsection that the Hamiltonians of the p -component modified KdV $_r$ hierarchy are defined by (23) up to some gauge transformation with fields u replaced by fields m using (40).

3.2 Dressing by formal series with respect to the spectral parameter

The goal of this subsection is to show that $p \times p$ matrix representation (31) in terms of the matrix differential operators of the r th order can be lifted to the $pr \times pr$ matrix representation of the differential operator of the first order depending on the spectral parameter λ . Our arguments here will be similar to the ones presented in [4].

We start from the initial linear problem for the n -size column $\overset{\circ}{\Psi}$

$$(\partial - \Lambda)\overset{\circ}{\Psi} = 0, \quad \Lambda = \Lambda_{p,r} \quad (42)$$

and define the formal $n \times n$ matrix series in λ by the equation

$$G(\Lambda) = \sum_{i=0}^{\infty} G_i \Lambda^{-i}, \quad (G_i)_{ab} = \begin{cases} \Delta^{r-a} C_{r-a}^{b-a} \partial_x^{b-a} K_i \Delta^{b-r} & b \geq a, \\ 0 & b < a, \end{cases} \quad (43)$$

where $a, b = 1, \dots, r$ and K_i are some $p \times p$ matrix function depending on all times $t_i^{(a)}$. Let $\Psi = G(\Lambda)\overset{\circ}{\Psi}$ and suppose that formal series $G(\Lambda)$ is subjected to the following constraints

$$G(\Lambda)(\partial_x - \Lambda)G^{-1}(\Lambda) = \partial_x - \Lambda + q^{\text{can}}, \quad (44)$$

$$G(\Lambda)(\partial_{t_s^{(a)}} - \Lambda_r^s \otimes E_a)G^{-1}(\Lambda) = \partial_{t_s^{(a)}} - V(\lambda) = \partial_{t_s^{(a)}} - \sum_{i=0}^s V_i \lambda^i. \quad (45)$$

These constraints show that Ψ is the simultaneous solution to the following linear problems

$$(\partial_x - \Lambda + q^{\text{can}})\Psi = 0, \quad (46)$$

$$(\partial_{t_s^{(a)}} - V(\lambda))\Psi = 0. \quad (47)$$

The following statement is valid.

Theorem 4. *The constraints (44) and (45) for the unknown functions K_i are equivalent to the equations (26) and (30) for the matrix Volterra operator $K = \mathbf{1}_p + \sum_{i=1}^{\infty} K_i \partial^i$. The zero coefficient with respect to λ of the matrix polynomial $V(\lambda)$ coincides with the expression (20) for the $\text{Grad}_q \mathcal{H}_{s,a}$.*

Proof. The equivalence of (44) and (26) follows from the straightforward calculations using (42) and the fact that $\Delta_i^r \neq \Delta_j^r$, $i \neq j$. To prove the equivalence of (45) and (30) we need the following technical lemma.

Lemma 5. *If $\Psi = (\psi_1, \dots, \psi_r)^t$ is the solution to the equation (46), where ψ_i are p -component vectors, then*

$$\partial_x^{-(i+1)} \psi_r = O(\lambda^{-1}) \psi_r + O(\lambda^{-1}) \psi_{r-1} + \dots + O(\lambda^{-1}) \psi_1, \quad i \geq 0. \quad (48)$$

Proof. The straightforward calculations similar to the ones presented in [4].

Now the statement about equivalence of (45) and (30) follows from the relation

$$\begin{aligned} V(\lambda)\Psi &= (G(\Lambda)_{t_s^{(a)}} + G(\Lambda)\Lambda_r^s \otimes E_a)\overset{\circ}{\Psi} = \\ &= \begin{pmatrix} \Delta^{r-1} \left[(K(\partial)_{t_s^{(a)}} K(\partial)^{-1} + K(\partial)(\Delta\partial)^s E_a K(\partial)^{-1}) \psi_r \right]^{(r-1)} \\ \Delta^{r-2} \left[(K(\partial)_{t_s^{(a)}} K(\partial)^{-1} + K(\partial)(\Delta\partial)^s E_a K(\partial)^{-1}) \psi_r \right]^{(r-2)} \\ \vdots \\ (K(\partial)_{t_s^{(a)}} K(\partial)^{-1} + K(\partial)(\Delta\partial)^s E_a K(\partial)^{-1}) \psi_r \end{pmatrix} \end{aligned} \quad (49)$$

and Lemma 5. Expression $[f]^{(i)}$ means here the i -fold derivative of the function f with respect to x . The absence of negative powers of ∂ in the operator $(K_{t_s^{(a)}} K^{-1} + K(\Delta\partial)^s E_a K^{-1})$ means the absence of negative powers of λ in the matrix $V(\lambda)$. Now the zero coefficient of the matrix $V(\lambda)$ with respect to λ can be obtained from (49) using $L\psi_r = K(\Delta\partial)^r K^{-1} \psi_r = \lambda \psi_r$ and explicit formulas (48) for $i = 1, \dots, r$ that follows from $\Psi = G(\Lambda)\overset{\circ}{\Psi}$ (see details in [4, 9]).

Let us lift up the discrete evolution (38) to the $n \times n$ matrix form. It is easy exercise to show using formulas (48) that absence of negative powers of λ in the operator ³

$$\left[G(t_0^{(a)} + 1)(\Lambda_r \otimes E_a)G(t_0^{(a)})^{-1} \right]_- = 0 \quad (50)$$

³Subscripts $-$ and $+$ in (50) and (51) mean the projections on the negative and nonnegative powers of the spectral parameter λ .

is equivalent to the discrete evolution of the matrix pseudo differential operator $K(x, \partial)$ (36). The equation (50) shows that dressed solution to the linear problems (46) and (47) satisfies also the discrete equation

$$\Psi(t_0^{(a)} + 1) = \left[G(t_0^{(a)} + 1)(\Lambda_r \otimes E_a)G(t_0^{(a)})^{-1} \right]_+ \Psi(t_0^{(a)}) \quad (51)$$

and p -component modified KdV $_r$ hierarchy can be obtained as compatibility condition of the linear problems (51) and (47). On the other hand the compatibility condition of the linear problems (51) and (46) yields the generalization of the Toda chain hierarchy. Indeed, we will see in the next subsection that this compatibility condition give exactly Toda chain hierarchy in the limit case $r = 1$. In fact, the p -component modified KdV $_r$ hierarchy which follows from (51) and (47) has inconvenient form, because our special choice of the dressing operator G . By this choice the fields matrix q^{can} in the operator $G(\Lambda)(\partial_x - \Lambda)G^{-1}(\Lambda)$ has the form (9) rather than block diagonal form and the mKdV $_r$ fields m_a enter to the operator $\left[G(t_0^{(a)} + 1)(\Lambda_r \otimes E_a)G(t_0^{(a)})^{-1} \right]_+$ in a very complicated and nonlinear way.

To write down the modified hierarchy in more convenient form we introduce new dressing operator $\tilde{G}(\Lambda)$ as the formal $n \times n$ matrix series with respect to the spectral parameter

$$\tilde{G}(\Lambda) = \sum_{i=0}^{\infty} \tilde{G}_i \Lambda^{-i}, \quad (\tilde{G}_i)_{ab} = \begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_r \end{pmatrix}, \quad (52)$$

where

$$H_i = \sum_{a=1}^p K_i(t_0^{(a)} + r - i)E_a \quad (53)$$

and K_i and H_i are $p \times p$ matrix functions. The theorem follows

Theorem 6. *The p -component modified KdV $_r$ hierarchy is defined by the compatibility condition of the linear problems for the column $\tilde{\Psi} = \tilde{G}(\Lambda)\overset{\circ}{\Psi}$*

$$(\partial - \Lambda + q^{\text{diag}})\tilde{\Psi} = 0, \quad (54)$$

$$(\partial_{t_s^{(a)}} - \tilde{V}(\lambda))\tilde{\Psi} = 0. \quad (55)$$

The discrete evolution

$$\tilde{\Psi}(t_0^{(a)} + 1) = \left[\tilde{G}(t_0^{(a)} + 1)(\Lambda_r \otimes E_a)\tilde{G}(t_0^{(a)})^{-1} \right]_+ \tilde{\Psi}(t_0^{(a)}) \quad (56)$$

is trivially satisfied due to (42) and block diagonal matrix q^{diag} is

$$q^{\text{diag}} = \text{diag}(M_r, \dots, M_1), \quad \text{where} \quad M_i = \Delta^{-1} \sum_{a=1}^p m_a(t_0^{(a)} + i - 1). \quad (57)$$

The matrix $\tilde{V}(\lambda)$ differs from the matrix $V(\lambda)$ by the gauge transformation which does not depend on the spectral parameter λ

$$\tilde{V}(\lambda) = S_{t_s^{(a)}} S^{-1} + S V(\lambda) S^{-1}, \quad (58)$$

where the $p \times p$ blocks of the matrix S are defined by ($1 \leq a, b \leq r$)

$$S_{ab} = \text{res} \left[\left(\sum_{c=1}^p ((\Delta \partial) E_c + m_c(t_0^{(c)} + r - a - 1) \cdots ((\Delta \partial) E_c + m_c(t_0^{(c)})) \right) \partial^{-r-1+b} \right]. \quad (59)$$

Proof. The proof is the direct calculation where the property of the matrices E_a

$$E_a E_b = \delta_{ab} E_a \quad (60)$$

is used [14]. Since the gauge transformation (58) does not depend on the spectral parameter the coefficient of $\tilde{V}(\lambda)$ at λ^0 can be easily obtained from (58).

3.3 Exceptional case of $r = 1$

In first section we have defined the regular elements of the loop algebra $\tilde{s}l_n$ for the equal block partition excluding the case $r = 1$ (pure homogeneous picture). In this subsection we will show that this case can be naturally included in Zakharov-Shabat dressing construction. The results presented here are well know from the early years of the development of the soliton theory (see [15] and reference therein) and we will present these results here for reader's convenience. The dressing condition (24) in this case is

$$L = K(\Delta \partial) K^{-1} = \Delta \partial + u, \quad \text{diag } u = 0 \quad (61)$$

and can be solved for the operator K due to $\Delta_i \neq \Delta_j$. The time evolution of the matrix u with respect to the continuum times $t_s^{(a)}$ and discrete times $t_0^{(a)}$ are defined by compatibility conditions of the following linear problems

$$L\psi = (K(\Delta \partial) K^{-1})\psi = (\Delta \partial + u)\psi = \lambda\psi, \quad (62)$$

$$\frac{\partial \psi}{\partial t_s^{(a)}} = (K E_a (\Delta \partial)^s K^{-1})_+ \psi, \quad (63)$$

$$\psi(t_0^{(a)} + 1) = (K(t_0^{(a)} + 1) E_a (\Delta \partial) K^{-1})_+ \psi = (E_a (\Delta \partial) + m_a) \psi. \quad (64)$$

The compatibility condition of (62) and (63) can be written in the form

$$\frac{\partial u_{ik}}{\partial t_s^{(a)}} = -\Delta_i \Delta_k \left(X_{ik}^{(1)} \right)' + \sum_{j=1}^p \left[\Delta_i X_{ij}^{(1)} u_{jk} - u_{ij} X_{jk}^{(1)} \Delta_k \right], \quad (65)$$

where matrix $X^{(1)}$ is defined by

$$X = \left(K(\Delta \partial)^{s-1} E_a K^{-1} \right)_- = \sum_{j=1}^{\infty} \partial^{-j} \circ X^{(j)}. \quad (66)$$

This hierarchy was called p -component NLS hierarchy in [16]. The equations (32) allows one to reconstruct the Hamiltonians corresponding to the hierarchy (65) using $X_{ij}^{(1)} = \delta \mathcal{H}_{s,a} / \delta u_{ij}$. It can be shown that diagonalization procedure (22) by means of the pseudo-differential

operator g yields the same Hamiltonians as diagonalization procedure due to Drinfeld and Sokolov [3] or the resolvent method introduced by Dickey [17]. The hierarchies of the type (65) were investigated in [18] for the case of arbitrary p and in [19] for the case $p = 2$.

When $s = 1$ we obtain equations

$$\begin{aligned}\frac{\partial u_{ik}}{\partial t_1^{(a)}} &= \frac{\Delta_a(\Delta_i - \Delta_k)}{(\Delta_a - \Delta_i)(\Delta_a - \Delta_k)} u_{ia} u_{ak}, \quad a \neq i, k, \\ \frac{\partial u_{ik}}{\partial t_1^{(i)}} &= \frac{\Delta_i \Delta_k}{\Delta_k - \Delta_i} u'_{ik} - \sum_{j \neq i} \frac{\Delta_i}{\Delta_j - \Delta_i} u_{ij} u_{jk}, \quad a = i, \\ \frac{\partial u_{ik}}{\partial t_1^{(k)}} &= -\frac{\Delta_i \Delta_k}{\Delta_k - \Delta_i} u'_{ik} + \sum_{j \neq k} \frac{\Delta_k}{\Delta_j - \Delta_k} u_{ij} u_{jk}, \quad a = k,\end{aligned}\tag{67}$$

which become trivial in the case $p = 2$.

For $s \geq 2$ the explicit form of the equation (65) becomes more complicated. Let us write down these equation for the simplest case $p = 2$. To do this we introduce new time variable t_2 (instead dependent due to (35) times $t_2^{(a)}$, $a = 1, 2$) such that

$$\frac{\partial L}{\partial t_2} = \frac{1}{i} \left(\frac{\partial L}{\partial t_2^{(1)}} - \frac{\partial L}{\partial t_2^{(2)}} \right)\tag{68}$$

and set $\Delta_1 = -\Delta_2 = 1$ for simplicity. In terms of this variable the equations (65) have most simplest form

$$i \frac{\partial u_{12}}{\partial t_2} = -\frac{1}{2} u''_{12} - u_{12}^2 u_{21}, \quad i \frac{\partial u_{21}}{\partial t_2} = \frac{1}{2} u''_{21} + u_{12} u_{21}^2,$$

and coincide with NLS equation after identification $u_{21} = \kappa \bar{u}_{12}$, where \bar{u}_{12} means the complex conjugation and $\kappa = \pm 1$.

The compatibility condition of (62) and (64) yields the multi-component analog of the Toda chain hierarchy. To see it, let us consider explicitly two-component case. Due to the relation (39) we can restrict ourself only to the evolution with respect to $t_0^{(1)}$. Now it is easy exercise to show that relation

$$(\Delta \partial + u(t_0^{(1)} + 1))(E_1 \Delta \partial + m_1(t_0^{(1)})) = (E_1 \Delta \partial + m_1(t_0^{(1)}))(\Delta \partial + u(t_0^{(1)}))\tag{69}$$

is equivalent to the first equation in the Toda chain hierarchy

$$\Delta_1 \Delta_2 \frac{\partial^2}{\partial x^2} \varphi(t_0^{(1)}) + e^{\varphi(t_0^{(1)}+1) - \varphi(t_0^{(1)})} - e^{\varphi(t_0^{(1)}) - \varphi(t_0^{(1)}-1)} = 0\tag{70}$$

and elements of the matrix u appear to be connected at the different values of the discrete time $t_0^{(1)}$

$$u_{12}(t_0^{(1)}) = u_{21}(t_0^{(1)} + 1)^{-1} = e^{\varphi(t_0^{(1)})}.\tag{71}$$

4 Conclusion

We have shown that Zakharov-Shabat dressing technique can be applied for the generalized integrable hierarchies discussed recently in [1, 7]. It allows one to conclude that there are exist

single dressing formalism that can be used for the description both principal (generalized KdV) and homogeneous (generalized AKNS) hierarchies as well as all intermediate ones.

To conclude let us outline the main steps of the construction. First, we started with a Hamiltonian reduction for the loop Lie algebra \tilde{g} admitting the regular element in the Heisenberg subalgebra. Solving the tangency constraints we have found an explicit expression for the gradient of the gauge invariant functional defined on the constrained manifold such that the corresponding Hamiltonian vector field ensures a tangency condition. Second, we developed a dressing technique in terms of pseudo-differential operators and formal series with respect to the spectral parameter and proved the coincidence of the integrable hierarchies which appeared in the framework of dressing approach with integrable hierarchies which emerged after Hamiltonian reduction on the reduced manifold \overline{M} .

All calculations have been carried out for the simplest case of the loop Lie algebra \tilde{sl}_n . It will be interesting to repeat these explicit calculations for the BCD and exceptional Lie algebras. In this case the first step of the construction can be easily repeated using ideas of the seminal paper [3]. To realize the second step one has to use the classification of the Heisenberg subalgebras admitting the regular elements which was obtained recently in [5]. It will be also interesting to investigate the relation between Hamiltonian structures of the BCD generalized Drinfeld-Sokolov hierarchies and the \mathcal{W} -algebras. These relations in case of affine Lie algebra \widehat{sl}_n were considered in [20]. The work in this direction is in progress.

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